Some new test cases in compressible thermo-fluid-dynamics

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This paper studies the thermo-fluid-dynamic field generated by a fast stream impinging on a plane. The Mach number is less than one, but not so small that compressibility effects can be neglected. The analysis requires first the nonviscous solution of the basic equation, obtained by assuming the velocity components as independent variables, and then the viscous solution in the boundary-layer approximation. The Dorodnitzin–Stewartson transformation is used to eliminate the dependence of the transport coefficients on the temperature. The values of 1 and 0.74 (air case) for the Prandtl number are considered. The viscous solution, obtained by means of a MacLaurin series and the Padé approximant technique, gives practically exact results.

Keywords: compressible boundary layer; heat transfer

Introduction

The exact analytical determination of a compressible flow requires the solution of the nonviscous equations and of the viscous boundary-layer equations. Both these systems of equations are difficult to solve, and thus it is difficult to obtain test cases for the compressible flow. In the incompressible case some useful exact solutions are available. In particular, the wedge flow has been extensively studied because it has applications of interest (its limiting cases give the flow along a flat plate and the plane stagnation flow) and is governed by very simple equations. In fact, the inviscid stream function is proportional to $r^{m+1} \sin(m + 1 - \pi)\theta$, where r and θ are polar coordinates, $m = \alpha/(\pi - \alpha)$, and α is the half-angle of the wedge. The viscous flow can be described in a similarity form and is governed by the Falkner–Skan equation (Schlichting 1968). These solutions hold only near the stagnation point, because as the inviscid velocity increases monotically along the wedge, at a certain distance from the origin of the axes the compressibility effects cannot be neglected.

To analyze the compressibility effects on the thermo-fluid-dynamic field, one must solve the coupled motion and energy equations both in the inviscid field and in the viscous one.

Forty years ago this problem was studied in an approximate form (Cohen and Reshotko 1956). Now we wish to determine some thermo-fluid-dynamic fields in an exact analytical form.

The compressible stream function ψ for the inviscid subsonic flow can be determined in the hodograph plane, i.e., by assuming the Cartesian component u and v of the velocity as independent variables. In this plane, the equation for ψ becomes linear and can be solved by separation of variables in terms of simple solutions (Von Mises and Geiringer 1958).

The boundary-layer equations can be written in a near incompressible form by means of the Dorodnitzin–Stewartson transformation.

The thermo-fluid-dynamic field has been found by means of expansions in series to be valid near the stagnation point. The Padé approximant technique has allowed us to obtain a representation valid in the entire field, following a procedure previously used by Pozzi and Lupo (1988) and Luchini, Lupo, and Pozzi (1990).

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Inviscid equations

By introducing the compressible stream function $\psi(\psi_y = \rho u/\rho_s$, $\psi_x = -\rho v/\rho_s$, where u and v are the velocity components along the Cartesian axes x and y and the subscript "s" indicates stagnation conditions), the two-dimensional (2-D) motion equations can be written as

$$\psi_{xx}(1 - u^2/a^2) - 2\psi_{xy}uv/a^2 + \psi_{yy}(1 - v^2/a^2) = 0 \tag{1}$$

$$a^2 + (\gamma - 1)V^2/2 = a_s^2 \quad \text{(Bernoulli equation)} \tag{2}$$

where a is the sound velocity, $V^2 = u^2 + v^2$, and γ is the ratio between the specific heat coefficients.

If one assumes V and θ, the angle between the velocity vector and the x-axis, as independent variables, Equation 2 becomes

$$V^2[1 - (\gamma - 1)V^2/(2a_s^2)]\psi_{VV} + [1 - (\gamma + 1)V^2/(2a_s^2)]\psi_{\theta\theta} + V[1 - (\gamma - 3)V^2/(2a_s^2)]\psi_V = 0 \tag{3}$$

It is convenient to introduce the variable $\tau = (\gamma - 1)V^2/(2a_s^2)$, where $a_s[2/(\gamma - 1)]^{1/2}$ is the limiting velocity V_L (the critical velocity V_C is related to V_L by the equation $V_C = V_L[(\gamma - 1)/(\gamma + 1)]^{1/2}$); therefore $V = a$ when $\tau = (\gamma - 1)/(\gamma + 1)$. In this way Equation 2 in a nondimensional form (a with respect to a_s) becomes

$$a^2 + \tau = 1 \tag{4}$$

and the density and the pressure can be written as

$$\rho/\rho_s = (1 - \tau)^{1/(\gamma - 1)}; \quad p/p_s = (1 - \tau)^{\gamma/(\gamma - 1)} \tag{5}$$

Solutions of Equation 3 can be written in the form

$$\psi_n = -\sin(n\theta + d_n)V^n f_n(\tau) \tag{6}$$

where f_n is the gaussian hypergeometric function $F(a_n, b_n, n + 1, \tau)$ with

$$a_n, b_n = \{(\gamma - 1)n - 1 \pm [(\gamma^2 - 1)n^2 + 1]\}/[2(\gamma - 1)]$$

For $\gamma = 1.4$ and $n = 2$, f_n becomes a polynomial: in this case, the stream function can be written in a nondimensional form as

$$\psi = -\sin 2\theta\, V^2 f_2(\tau)/2 \tag{7}$$

where $f_2 = 1 - 5\tau/2 + 35\tau^2/16 - 21\tau^3/32$ and V is nondimensionalized with respect to a suitable reference velocity V_r (see Appendix): therefore it is $\tau = k^2V^2$, where $k = V_r/V_L$.

This function vanishes when $\theta = 0$ and $\theta = \pi/2$, and therefore it represents the stream function of a flow symmetrical with respect to the x-axis impinging on a plane: the inflow and the outflow can be obtained from Equation 7. The last streamline that we consider can be assumed to represent a wall.

The potential function can be determined from the irrotationality equation $\varphi_\theta = V\psi_V\rho_s/\rho$. One has

$$\varphi = V^2(1 - 5\tau + 105\tau^2/16 - 21\tau^3/8)\cos 2\theta/2(1 - \tau)^{5/2}$$

The expressions giving $x(V, \theta)$ and $y(V, \theta)$ can be obtained from the following equations:

$$x_V = (\varphi_V \cos\theta - \sin\theta\psi_V\rho_s/\rho)/V$$
$$y_V = (\varphi_V \sin\theta + \cos\theta\psi_V\rho_s/\rho)/V \tag{8}$$

or

$$x_\theta = (\varphi_\theta \cos\theta - \sin\theta\psi_\theta\rho_s/\rho)/V$$
$$y_\theta = (\varphi_\theta \sin\theta + \cos\theta\psi_\theta\rho_s/\rho)/V \tag{9}$$

From Equation 9 in nondimensional form, one has

$$x = V\cos\theta\left[3 - \frac{15}{2}\tau + \frac{105}{16}\tau^2 - \frac{63}{32}\tau^3 - \cos^2\theta\left(5\tau - \frac{35}{4}\tau^2 + \frac{63}{16}\tau^3\right)\right]\Big/3(1 - \tau)^{5/2} \tag{10a}$$

$$y = -V\sin\theta\left[3 - \frac{15}{2}\tau + \frac{105}{16}\tau^2 - \frac{63}{32}\tau^3 - \sin^2\theta\left(5\tau - \frac{35}{4}\tau^2 + \frac{63}{16}\tau^3\right)\right]\Big/3(1 - \tau)^{5/2} \tag{10b}$$

and in particular, from Equation 10a, it is

$$x_W = V(3 - 25\tau/2 + 245\tau^2/16 - 189\tau^3/32)/3(1 - \tau)^{5/2} \tag{11}$$

where the subscript "w" indicates the wall.

Boundary-layer equations

The boundary-layer equations in a nondimensional form can be written as

$$(\rho u)_x + (\rho v)_y = 0 \tag{12}$$

$$\rho(uu_x + vu_y) = \rho_e u_e u_{ex} + (\mu u_y)_y \tag{13}$$

$$\rho(uS_x + vS_y) = (\lambda S_y)_y/\mathrm{Pr} + (\mathrm{Pr} - 1)[\mu V_r^2 uu_y]_y/(\mathrm{Pr}\, H_e) \tag{14}$$

where the subscript "e" indicates the external inviscid condition calculated on the wall; μ and λ are the viscosity and thermal conductivity coefficients, respectively, and $S = h_{tot}/h_{tot,e} - 1$ and h_{tot} is the total enthalpy $h + V^2/2$. The boundary conditions associated with Equations 12–14 are $u(x, 0) = v(x, 0) = 0$; $u(x, \infty) = u_e$; $S(x, 0) = S_w$, $S(x, \infty) = 0$.

By means of the Dorodnitzin–Stewartson transformation $X = \int_0^x a_e^m\, dx$; $Y = a_e \int_0^y \rho\, dy$; $U = u/a_e$; $V = (UY_x + \rho v)/a_e^m$, where $m = (3\gamma - 1)/(\gamma - 1)$ and a_e and ρ are nondimensionalized with respect to respective stagnation quantities, Equations 12 and 13 become

$$U_X + V_Y = 0 \tag{15}$$

$$UU_X + VU_Y = U_{YY} + (1 + S)U_eU_{eX} \tag{16}$$

$$US_X + VS_Y = S_{YY}/\mathrm{Pr} + (\mathrm{Pr} - 1)a^2V_r^2UU_y/(\mathrm{Pr}\, H_e) \tag{17}$$

Notation

a	Sound velocity
f_i	Function defined by Equations 20
f_n	Function defined by Equation 6
f_2	Function defined by Equation 7
h	Enthalpy
k	V_r/V_L
m	$(3\gamma - 1)/(\gamma - 1)$
p	Pressure
Pr	Prandtl number
S	$h_{tot}/h_{tot,e} - 1$
S_i	Functions defined by Equations 20
u, v	Components of velocity along x- and y-axes
U, V	$U = u/a_e$, $V = (UY_x + \rho v)/a_e^m$ (Dorodnitzin–Stewartson variables)
V	Modulus of velocity ($V^2 = u^2 + v^2$)
x, y	Cartesian axes
X, Y	Dorodnitzin–Stewartson coordinates

Greek symbols

γ	Ratio between the specific heat coefficients
θ	Angle between the velocity vector and x-axis
λ	Thermal conductivity
μ	Viscosity
ρ	Density
τ	V^2/V_L^2
φ	Potential function
ψ	Stream function ($\rho_s\psi_y = \rho u$, $\rho_s\psi_x = -\rho v$)

Subscripts

c	Critical
e	External
L	Limiting
r	Reference
s	Stagnation
tot	Total
w	Wall

By introducing the stream function ψ ($\psi_Y = U$ and $\psi_X = -V$) and the variable $\tau(X)$ given by Equation 11, Equations 16 and 17 can be written as

$$\psi_Y \psi_{Y\tau} - \psi_\tau \psi_{YY} = \psi_{YYY} X_\tau + (1 + S) U_e U_{e\tau} \tag{18}$$

$$\psi_Y S_\tau - \psi_\tau S_Y = S_{YY} X_\tau/\mathrm{Pr} + (\mathrm{Pr} - 1)(1 - \tau) X_\tau \psi_y \psi_{yy} V_\tau^2/(\mathrm{Pr}\, H_e) \tag{19}$$

These equations can be solved by putting

$$k\psi = [\tau/(1 - \tau)]^{1/2} \sum_{i=0}^{\infty} \tau^i f_i(Y); \quad S = \sum_{i=0}^{\infty} \tau^i S_i(Y) \tag{20}$$

with $f_i(0) = f_i'(0) = 0$; $f_0'(\infty) = 1$; $f_i'(\infty) = 0$ for $i > 0$; $S_0(0) = S_w$; $S_i(0) = 0$ for $i > 0$ and $S_i(\infty) = 0$.

By substituting Equation 20 into Equations 18 and 19, one has the equations for the unknowns f_i and S_i. In particular, the leading order of the expansion gives

$$\begin{aligned} &f_0''' + f_0'' f_0 - f_0'^2 = -1 - S_0 \\ &S_0''/\mathrm{Pr} + f_0 S_0' = 0 \end{aligned} \tag{21}$$

while f_n and S_n, with $n > 0$, are given by

$$\begin{aligned} &f_n''' + f_0 f_n'' - 2(n+1) f_0' f_n' + (2n+1) f_0'' f_n = -S_n + F_n \\ &S_n''/\mathrm{Pr} + f_0 S_n' - 2n S_n f_0' = G_n \end{aligned} \tag{22}$$

where

$$g(\tau) = (1 - 17\tau/2 + 275\tau^2/16 - 441\tau^3/32 + 63\tau^4/16)/2 = \sum g_i \tau^i$$

$$\begin{aligned} F_n = {} & -2f_0''(g_n - 2g_{n-1} + g_{n-2}) \\ & + 2(n-1) f_0'' f_{n-1} - 2(n-1) f_0' f_{n-1}' \\ & - 2\sum_{i=1}^{n-i} \{ f_{n-i}'''(g_i - 2g_{i-1} + g_{i-2}) + f_{n-i}''[(i + 1/2) f_i \\ & - (i-1) f_{i-1} - f_{n-i}'[(i + 1/2) f_i' - (i-1) f_{i-1}'] \} \end{aligned}$$

$$\begin{aligned} G_n = {} & -2 \sum_{i=1}^{n} \{ S_{n-i}''(g_i - 2g_{i-1} + g_{i-2}) \\ & + (n-i) S_{n-i}(f_i' - f_{i-1}') \\ & - S_{n-i}'[f_i(i + 1/2) - (i-1) f_{i-1}] \} - 2(\mathrm{Pr} - 1) h_n/\mathrm{Pr} \end{aligned}$$

where

$$h_n = \sum_{i=0}^{n} \left(\sum_{j=0}^{n-i} f_j' f_{n-i-j}'' \right) (g_{i-1} - 2g_{i-2} + g_{i-3})$$

The representation of ψ and S given by Equation 20 is not valid for any value of τ. In order to evaluate the range of validity of such expansion and to obtain a representation valid for higher values of τ, we use the Padé representation (Bender and Orsag 1959) by putting a function $f = \Sigma \tau^i f_i$ in the rational form $P_n(\tau)/Q_n(\tau)$, where P_n and Q_n are polynomials of degree n whose coefficients are determined from the equation $\Sigma \tau^i f_i = P_n/Q_n$, and by imposing that such an equation is satisfied up to terms of order of τ^{2n}. In particular, if one writes $P_n = \sum_{i=0}^n A_i \tau^i$ and $Q_n = \sum_{i=0}^n B_i \tau^i$, one has $(\Sigma A_i \tau^i)(\Sigma f_i \tau^i) = \Sigma B_i \tau^i$; by equating the coefficients of the same powers of τ, one has $B_i = \sum_{j=0}^i A_j f_{i-j}$. These equations must be written for $i = 0, 1, 2 \ldots 2n$ and must give the $2n + 1$ unknowns. The radius of convergence r of the expansion (Equations 20) is given by the root of $Q_n = 0$ having the smallest modulus: such a modulus is an estimation of r. The Padé approximants also gives a representation of ψ and S valid when the expansion (Equation 20) does not converge.

Analysis of the results

The inviscid compressible fluid-dynamic field we have studied is described by Equations 7–11.

The streamlines of such a flow are drawn in Figure 1. Any two of these streamlines can be considered as the walls of a duct discharging against a plane plate. The nondimensional modulus of velocity, V, and its inclination θ with respect to the x-axis at $y = -1$ are plotted in Figure 2: in this way, the initial conditions of this flow are known. V and θ at $x = 0.2$ and $x = 0.5$ are plotted in Figure 3. Figure 4 gives Kx versus τ.

The nonviscous solution gives the velocity U_e on the plate, through Equation 11, that allows us to solve the boundary-layer equations by means of the expansion (Equations 20). The numerical solution of Equations 21 and 22, found using the Runge Kutta method, has been obtained by considering 13 terms of Equations 20. The radius of convergence r of Mac–Laurin expansion has been determined by the Padé

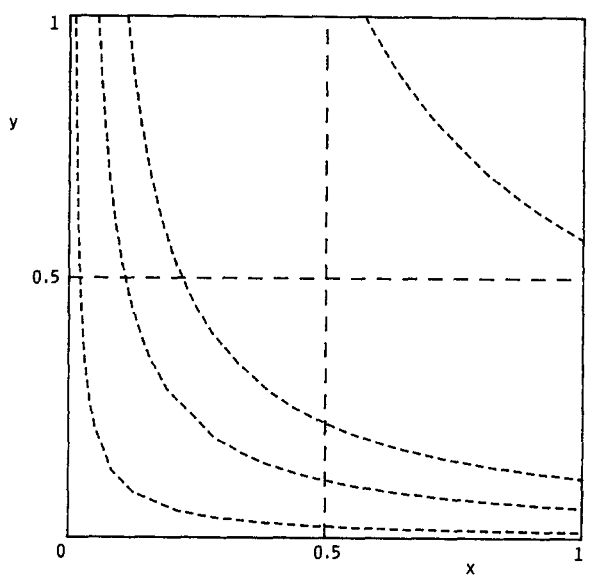

Figure 1 Streamline; $\psi = 0.01, 0.05, 0.1, 0.5$

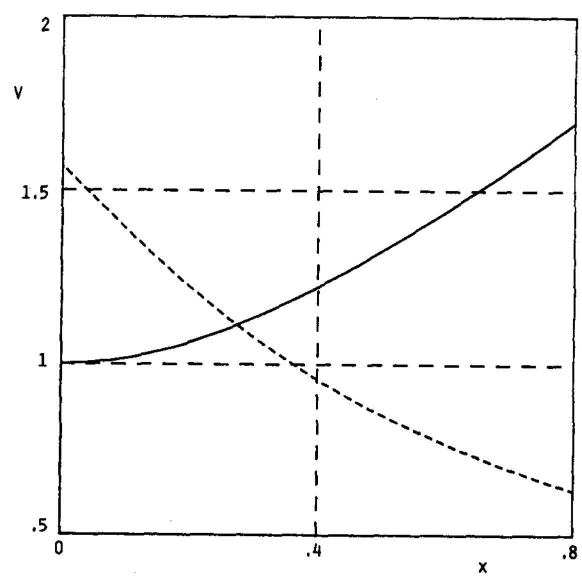

Figure 2 $V(x)$ — and $\theta(x)$ --- at $y = -1$

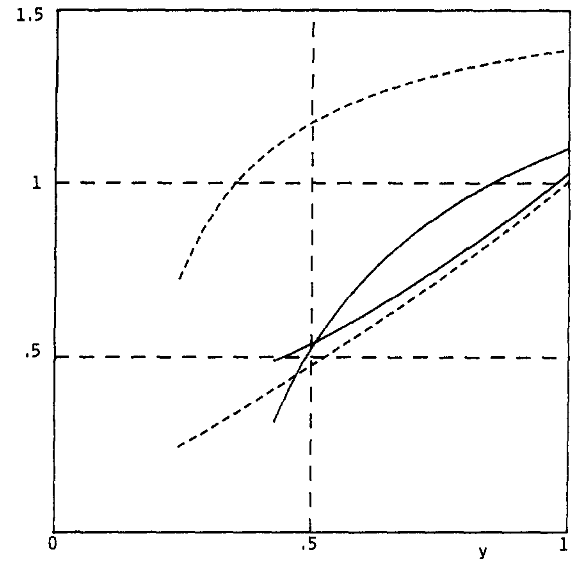

Figure 3 $V(y)$ and $\theta(y)$ at $x = 0.2$ --- and $x = 0.5$ —

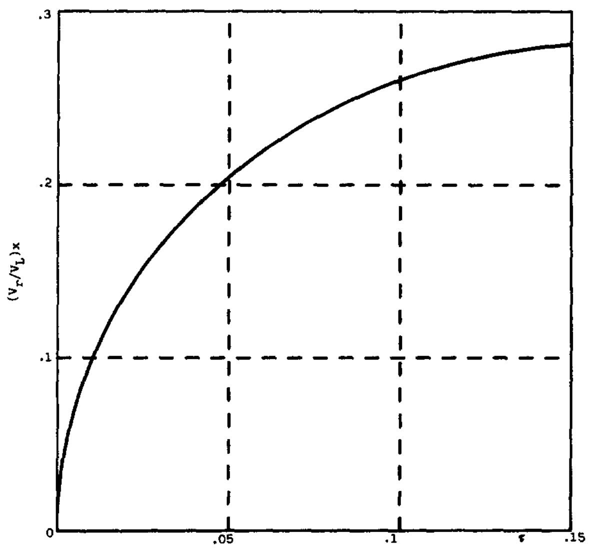

Figure 4 Abscissa versus velocity

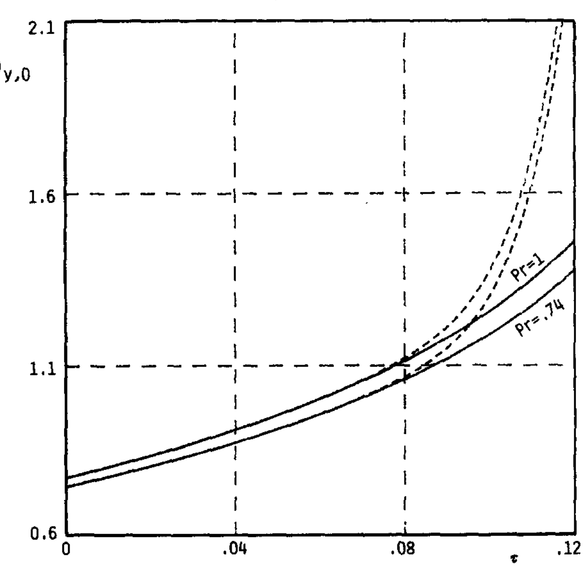

Figure 5 Comparison between MacLaurin expansion --- $\Sigma f_i''\tau^i$ and Padé approximant — of $(u/U_e)_{Y,0}$ for Pr = 1, Pr = 0.74, and $S_w = -0.8$

approximants technique: we found $r = 0.06$. The Padé approximants also allow to obtain a representation of the functions ψ and S valid for values of $\tau > r$.

The second derivative of $\Sigma f_i \tau^i$ and the first of $\Sigma S_i \tau^i$ at $y = 0$ are drawn in Figures 5, 6, 7, and 8; dashed curves represent the expansion (Equations 20), while solid curves represent Padé approximants. The curves of $u_{y,0}$ and S are drawn for Pr = 1 and Pr = 0.74, both for $S_w = -0.8$ and for $S_w = -0.4$. We can see that the two representations practically coincide when $\tau < r$; when $\tau > r$, the Mac–Laurin series diverges while the Padé representation is regular. Nondimensional shear stress $u_{Y,0}$ and heat flux $S_{Y,0}$ are therefore exactly given by the Padé approximants in the entire field of interest. The same quantities are drawn (for $M = 0.5$ and $M = 0.75$) versus x in Figures 9 and 10.

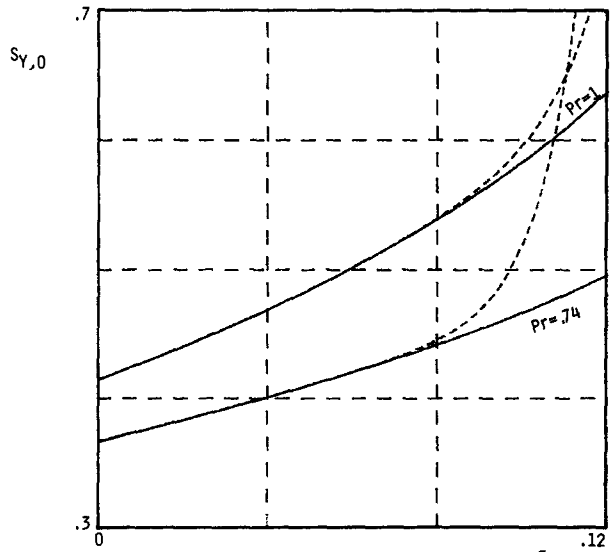

Figure 6 Comparison between MacLaurin expansion --- $\Sigma S_i'\tau^i$ and Padé approximant — of $S_{Y,0}$ for Pr = 1, Pr = 0.74 and $S_w = 0.8$

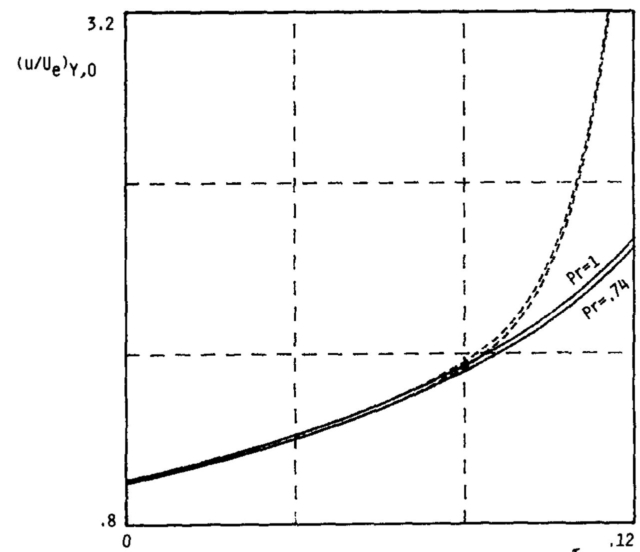

Figure 7 Comparison between MacLaurin expansion --- $\Sigma f_i''\tau^i$ and Padé approximant — of $(u/U_e)_{Y,0}$ for Pr = 1, Pr = 0.74, and $S_w = -0.4$

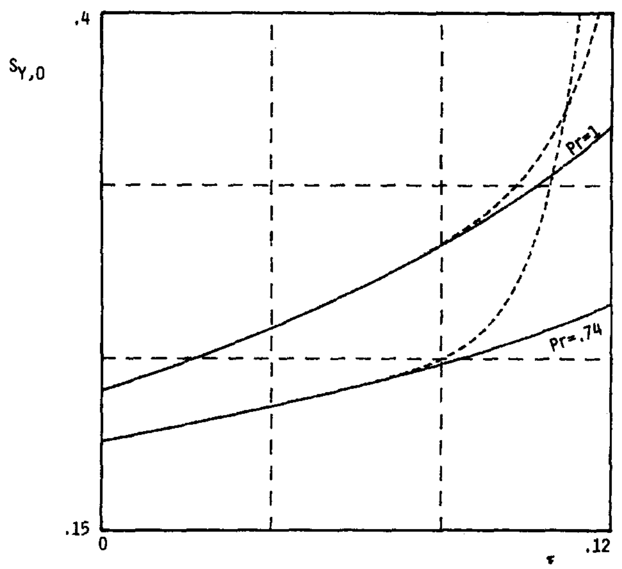

Figure 8 Comparison between MacLaurin expansion --- $\Sigma S_i' \tau^i$ and Padé approximant — of $S_{Y,0}$ for Pr = 1, Pr = 0.74, and $S_w = -0.4$

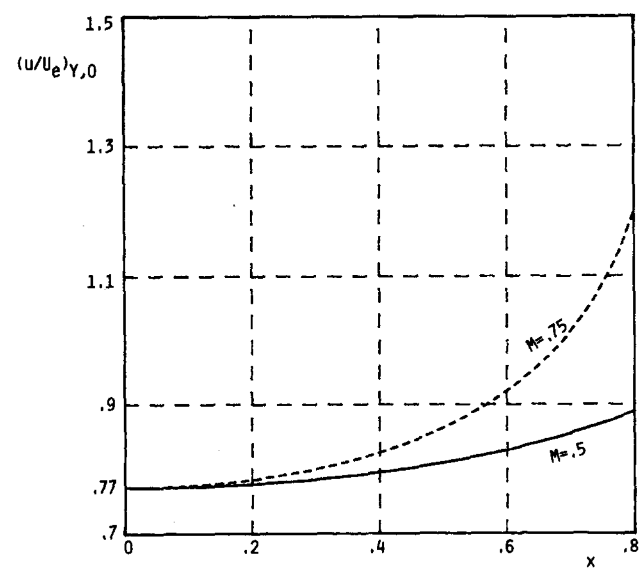

Figure 9 $(u/U_e)_{Y,0}$ versus x for $M = 0.5$ and $M = 0.75$, Pr = 1, $S_w = -0.8$

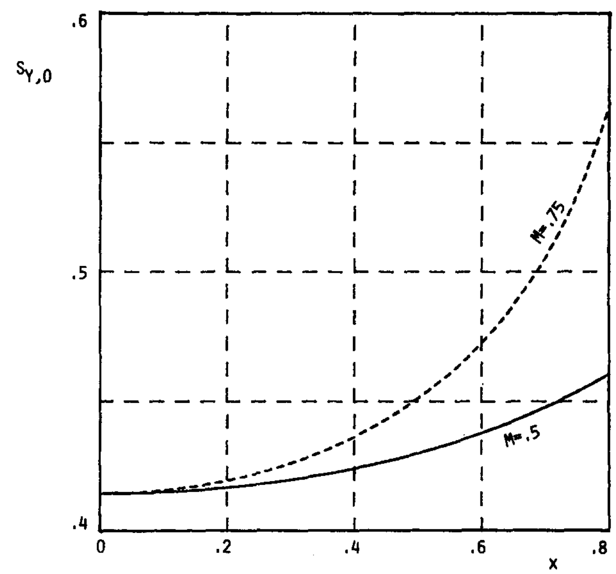

Figure 10 $S_{Y,0}$ versus x for $M = 0.5$ and $M = 0.75$, Pr = 1, $S_w = -0.8$

Concluding remarks

In this paper, we have presented the solution of a compressible thermo-fluid-dynamic field generated by a stream that impinges on a plane plate. The nonviscous solution of this problem has been found in a simple form in the hodograph plane. The viscous solution has been found by using the Stewartson–Dorodnitzin transformation and by expanding the stream function and the total enthalpy in terms of the variable $\tau = V^2/V_L^2$. The leading terms of such expansion, given by Equations 21, describe the thermo-fluid-dynamic flow, in the transformed plane, that occurs when $U_e = X$ and terms of order X^2 in Equation 17 are neglected. The value of the radius of convergence of the expansion is 0.06; therefore the expansion (Equations 20) holds for $0 \leq V/V_L < 0.245$. In order to obtain the solution for higher values of V, we have introduced the Padé approximation: in this way, accurate results are available for any subsonic value of V.

Acknowledgment

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Appendix: Reference quantities

The reference density is the stagnation one (ρ_s).

The reference length is the distance L between point B, which represents the beginning of the equivalent duct, and the wall.

The reference stream function ψ_r is given in terms of the mass rate flow $\dot{m} = (\psi_{ad})_B\, \rho_s\, \psi_r$ (T_s and p_s can be written in terms of V_L and ρ_s as: $T_s = (\gamma - 1)V_L^2/2\gamma R$, $p_s = (\gamma - 1)\rho_s V_L^2/2\gamma$).

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